



# High-order numerical method for generalized Black-Scholes model

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## Abstract

This work presents a high order numerical method for the solution of generalized Black-Scholes model for European call option. The numerical method is derived using a two-step backward differentiation formula in the temporal discretization and a High-Order Difference approximation with Identity Expansion (HODIE) scheme in the spatial discretization. The present scheme gives second order accuracy in time and third order accuracy in space. Numerical experiments are conducted in support of the theoretical results.

**Keywords:** Generalized Black-Scholes model, Two-step backward differentiation formula, HODIE scheme

## 1 Introduction

Option pricing is a well flourished subject in the financial market. Options are the tools against the uncertainty of the market. The writer of an option gives its holder right to hedge the risk by limiting the loss, for which the writer is paid a premium called the option price ([29, 2]).

Let there be a frictionless and arbitrage free financial market comprised of a risk-free asset and a unit risky asset. The following stochastic differential equation

$$dS = (\mu - D)Sd\tau + \sigma SdW, \quad (1)$$

gives a mathematical representation for the asset price  $S$  of the unit risky asset. Here  $\mu$  is the drift rate,  $D$  is the dividend yield,  $\sigma$  is the market volatility and  $dW$  is the increment of a standard Wiener process. The following famous Black-Scholes equation for evaluating the option price  $C(S, \tau)$ , is derived applying Itô's lemma and through elimination of randomness in a complete market

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S \frac{\partial C}{\partial S} - rC = -\frac{\partial C}{\partial \tau}, \quad S > 0, \tau \in (0, T), \quad (2)$$

along with the final condition

$$C(S, T) = \max(S - K, 0), \quad S \in [0, \infty), \quad (3)$$

where  $\tau$  is the time variable,  $r$  is the risk free interest rate,  $T$  is the expiry time of the contract and  $K$  is the strike price. It was derived by Fischer Black and Myron Scholes in their seminal work [1].

When  $\sigma$ ,  $r$  and  $D$  are constants, the Black-Scholes equation can be easily transformed into standard heat equation, and the following analytical formula is obtained for pricing the option ([1, 30, 21]).

$$C(S, \tau) = S \exp(-D(T - \tau))N(d_1) - K \exp(-r(T - \tau))N(d_2), \quad (4a)$$

where

$$d_1 = \frac{\ln S - \ln K + (r - D + \frac{1}{2}\sigma^2)(T - \tau)}{\sigma\sqrt{T - \tau}}, \quad d_2 = d_1 - \sigma\sqrt{T - \tau} \quad (4b)$$

and  $N(y)$  is the cumulative standard normal distribution function given by

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \exp\left(-\frac{1}{2}x^2\right) dx. \quad (4c)$$

But in the financial market the parameters  $\sigma$ ,  $r$  and  $D$  can be dependent on the asset price  $S$  and the time variable  $\tau$ . Analytical solution of such generalized Black-Scholes model is not known.

The generalized Black-Scholes model was numerically solved in [14, 22, 25, 28, 13, 11, 4, 3]. Cubic B-spline collocation method was used in [14, 13] to get a second order scheme for approximating generalized Black-Scholes model. In [11], cubic polynomial spline method and implicit Euler method was used in space and time respectively to solve the model, achieving second order accuracy in space. Cubic B-spline collocation method and Crank-Nicolson time-stepping scheme was used in space and time direction respectively in [14] and [13] to get second order accuracy in both space and time for the generalized model. In the work [22], the author achieved third order accuracy in space using the quintic B-spline collocation method and first and second order accuracy in time using backward Euler method and Crank-Nicolson method respectively. A novel fitted finite volume method along with an implicit time stepping technique was used in [28] and an  $O(h)$  accuracy was achieved. In the work [25], a fitted finite volume method was applied to the generalized model with  $O(h^2)$  order of convergence. In [4], the authors have achieved second order accuracy using an exponential time integration scheme along with a central difference scheme for spatial discretization. A method based on central difference spatial discretization and implicit time stepping technique was applied in [3] to achieve  $O(k + h^2)$  accuracy. Lattice approach was also applied for option pricing, such as binomial formula, also called CRR model was developed in [8], a trinomial process was introduced in [23] and such lattice approach was further generalized in [12].

In the present paper we construct a high order numerical scheme for approximating the option price governed by the generalized Black-Scholes model for European call option. We have applied two-step backward differentiation formula in temporal discretization to achieve second order convergence in time direction and the classical High-Order Difference approximation with Identity Expansion (HODIE) scheme where three nodal auxiliary points are used for identity expansion to achieve third convergence in spatial direction. We have done simultaneous discretization in time and space directions and hence established the convergence of the scheme

with simpler arguments ([7]). This yields a finite difference scheme which has second order convergence in time and third order convergence in space.

The rest of the paper is organized as follows: Section 2 describes the generalized Black-Scholes model. Smoothing of the terminal condition, some transformations and the truncation of the infinite domain for numerical purpose are given. In Section 3, simultaneous discretization of the parabolic partial differential equation along with initial and boundary conditions is done and the numerical scheme is derived. In Section 4, convergence analysis of the numerical scheme is done. Numerical experiments are given in Section 5 and Section 6 concludes the paper.

## 2 The generalized Black-Scholes model

The generalized Black-Scholes model for pricing European call option is

$$-\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2(S, \tau)S^2\frac{\partial^2 C}{\partial S^2} + (r(S, \tau) - D(S, \tau))S\frac{\partial C}{\partial S} - r(S, \tau)C, \quad S > 0, \tau \in (0, T), \quad (5a)$$

along with the final condition

$$C(S, T) = \max(S - K, 0), \quad S \in [0, \infty) \quad (5b)$$

and boundary conditions

$$C(0, \tau) = 0, \quad (5c)$$

$$C(S, \tau) \rightarrow S, \quad \text{as } S \rightarrow \infty. \quad (5d)$$

Here  $\sigma$ ,  $r$  and  $D$  are assumed to be continuous and bounded functions.

The terminal condition of the above problem is not smooth. Hence to ensure the convergence of the numerical solution ([10, 24]), we replace a small  $\epsilon$ -neighbourhood of the point of singularity by a eleventh degree polynomial  $\phi(S)$  so that the payoff is a fifth order smooth function ([3]). For getting higher order accurate numerical scheme, we apply the transformations  $S = \exp(x)$  and  $\tau = T - t$ , where  $x$  and  $t$  are the new space and time variables respectively, to change our degenerate final boundary value problem (5) into a non-degenerate initial boundary value problem. We will give new notation to all the variables, for example any variable say  $X(S, \tau)$  is now  $\hat{X}(x, t)$ .

To apply the numerical scheme, we truncate the infinite space domain  $(-\infty, \infty)$  and take the domain  $[x_{min}, x_{max}]$ , where  $x_{min}$  and  $x_{max}$  and the boundary conditions at left and right boundaries are chosen suitably ([27, 15]). The existence and uniqueness of the problem considered and of its various intermediate forms during the stated modifications are well established ([9, 5, 18]).

Define the domain  $\Omega = (x_{min}, x_{max}) \times (0, T]$ . After the stated modifications, we have the following initial boundary value problem which we will solve numerically

$$Lu(x, t) \equiv \frac{\partial u}{\partial t} - a_2(x, t)\frac{\partial^2 u}{\partial x^2} - a_1(x, t)\frac{\partial u}{\partial x} - a_0(x, t)u = f(x, t), \quad (x, t) \in \Omega \quad (6a)$$

where

$$\begin{aligned} a_2(x, t) &= \frac{1}{2}\hat{\sigma}^2(x, t), \\ a_1(x, t) &= \hat{r}(x, t) - \hat{D}(x, t) - \frac{1}{2}\hat{\sigma}^2(x, t), \\ a_0(x, t) &= -\hat{r}(x, t), \end{aligned}$$

with the initial condition

$$u(x, 0) = \phi(x), \quad x \in [x_{min}, x_{max}], \quad (6b)$$

and the boundary conditions

$$\hat{g}_1(t) \equiv u(x_{min}, t) = 0, \quad t \in [0, T],$$

$$\hat{g}_2(t) \equiv u(x_{max}, t) = \exp \left( x_{max} - \int_0^t \hat{D}(x_{max}, q) dq \right) - K \exp \left( - \int_0^t \hat{r}(x_{max}, q) dq \right), \quad t \in [0, T].$$

where  $\hat{g}_1$  and  $\hat{g}_2$  are the left and the right boundary conditions.

Let us assume that  $a_2(x, t) \geq \mu > 0$ ,  $a_1(x, t) \geq \nu > 0$  and  $-a_0(x, t) \geq \gamma > 0$ . Sufficient regularity conditions and the following compatibility conditions

$$\phi(x_{min}) = \hat{g}_1(0) \quad \text{and} \quad \phi(x_{max}) = \hat{g}_2(0), \quad (7)$$

are also assumed, which ensures the existence and uniqueness of the solution  $u \in C(\bar{\Omega})C^{2,1}(\Omega)$  ([18, 9]) satisfying

$$\left| \frac{\partial^{m+n} u}{\partial x^m \partial t^n}(x, t) \right| \leq \mathcal{C} \text{ on } \bar{\Omega}; \quad 0 \leq n \leq 3 \text{ and } 0 \leq m + n \leq 5. \quad (8)$$

Note that  $\mathcal{C}$  is a generic constant.

### 3 Discretization

We discretize the domain  $\bar{\Omega}$  uniformly in time and space direction. Let  $(x_m, t_n)$ ,  $m = 0, 1, \dots, M$ ,  $n = 0, 1, \dots, N$  be the grid points in the discretized domain  $\bar{\Omega}_h^k$ , where  $M$  is the total number of intervals in space direction,  $N$  is the total number of intervals in time direction,  $h = x_{m+1} - x_m$ ,  $m = 0, 1, \dots, M-1$  and  $k = t_{n+1} - t_n$ ,  $n = 0, 1, \dots, N-1$ . The fully discrete scheme on this mesh is given by

$$\begin{aligned} \mathcal{L}_h^k \equiv & \beta_{m,1}^n (\delta_t U_{m-1}^n) + \beta_{m,2}^n (\delta_t U_m^n) + \beta_{m,3}^n (\delta_t U_{m+1}^n) + [\alpha_{m,-}^n U_{m-1}^n + \alpha_{m,c}^n U_m^n + \alpha_{m,+}^n U_{m+1}^n] = \\ & \beta_{m,1}^n f_{m-1}^n + \beta_{m,2}^n f_m^n + \beta_{m,3}^n f_{m+1}^n, \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N, \end{aligned} \quad (9a)$$

where

$$\delta_t U_m^n = (U_m^n - U_m^{n-1})/k, \quad n = 1,$$

$$\delta_t U_m^n = \left( \frac{3}{2} U_m^n - 2 U_m^{n-1} + \frac{1}{2} U_m^{n-2} \right) / k, \quad n = 2, 3, \dots, N,$$

$$U_m^0 = \phi_m, \quad m = 0, 1, \dots, M, \quad (9b)$$

$$U_0^n = \hat{g}_1^n, \quad n = 0, 1, \dots, N, \quad (9c)$$

$$U_M^n = \hat{g}_2^n, \quad n = 0, 1, \dots, N. \quad (9d)$$

Here we have applied two-step backward differentiation method in time direction using backward Euler's formula for the first time level. Space direction is discretized using classical High-Order

Difference approximation with Identity Expansion (HODIE) scheme ([20, 6]). We have taken three auxiliary points  $x_{m-1}, x_m, x_{m+1}$ . The HODIE coefficients  $\alpha's$  and  $\beta's$ , are computed “locally” by making (9a) exact on  $P_4$ , the space of polynomials of degree less than or equal to 4. In order to compute these coefficients uniquely we use the following normalization condition

$$\beta_{m,1}^n + \beta_{m,2}^n + \beta_{m,3}^n = 1, \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N. \quad (10)$$

Now the HODIE equations are

$$\begin{aligned} \alpha_{m,-}^n + \alpha_{m,c}^n + \alpha_{m,+}^n &= \beta_{m,1}^n(-a_{0,m-1}^n) + \beta_{m,2}^n(-a_{0,m}^n) + \beta_{m,3}^n(-a_{0,m+1}^n), \\ -h\alpha_{m,-}^n + h\alpha_{m,+}^n &= \beta_{m,1}^n(-a_{1,m-1}^n + ha_{0,m-1}^n) + \beta_{m,2}^n(-a_{1,m}^n) \\ &\quad + \beta_{m,3}^n(-a_{1,m+1}^n - ha_{0,m+1}^n), \\ h^2\alpha_{m,-}^n + h^2\alpha_{m,+}^n &= \beta_{m,1}^n(-2a_{2,m-1}^n + 2ha_{1,m-1}^n - h^2a_{0,m-1}^n) + \beta_{m,2}^n(-2a_{2,m}^n) \\ &\quad + \beta_{m,3}^n(-2a_{2,m+1}^n - 2ha_{1,m+1}^n - h^2a_{0,m+1}^n), \\ -h^3\alpha_{m,-}^n + h^3\alpha_{m,+}^n &= \beta_{m,1}^n(6ha_{2,m-1}^n - 3h^2a_{1,m-1}^n + h^3a_{0,m-1}^n) \\ &\quad + \beta_{m,3}^n(-6ha_{2,m+1}^n - 3h^2a_{1,m+1}^n - h^3a_{0,m+1}^n), \\ h^4\alpha_{m,-}^n + h^4\alpha_{m,+}^n &= \beta_{m,1}^n(-12h^2a_{2,m-1}^n + 4h^3a_{1,m-1}^n - h^4a_{0,m-1}^n) \\ &\quad + \beta_{m,3}^n(-12h^2a_{2,m+1}^n - 4h^3a_{1,m+1}^n - h^4a_{0,m+1}^n) \end{aligned}$$

for  $m = 1, 2, \dots, M-1$  and  $n = 1, 2, \dots, N$ . Solving the above HODIE equations along with the normalization condition (10), we get the HODIE coefficients

$$\alpha_{m,-}^n = \frac{1}{2h^2}[\beta_{m,1}^n(-2a_{2,m-1}^n + 3ha_{1,m-1}^n - 2h^2a_{0,m-1}^n) + \beta_{m,2}^n(-2a_{2,m}^n + ha_{1,m}^n) + \beta_{m,3}^n(-2a_{2,m+1}^n - ha_{1,m+1}^n)], \quad (11)$$

$$\alpha_{m,+}^n = \frac{1}{2h^2}[\beta_{m,1}^n(-2a_{2,m-1}^n + ha_{1,m-1}^n) + \beta_{m,2}^n(-2a_{2,m}^n - ha_{1,m}^n) + \beta_{m,3}^n(-2a_{2,m+1}^n - 3ha_{1,m+1}^n - 2h^2a_{0,m+1}^n)] \quad (12)$$

and

$$\alpha_{m,c}^n = -(\beta_{m,1}^na_{0,m-1}^n + \beta_{m,2}^na_{0,m}^n + \beta_{m,3}^na_{0,m+1}^n + \alpha_{m,-}^n + \alpha_{m,+}^n). \quad (13)$$

Then final scheme becomes

$$\begin{aligned} L_h^k U_m^n &\equiv \left(\alpha_{m,-}^n + \frac{3}{2k}\beta_{m,1}^n\right) U_{m-1}^n + \left(\alpha_{m,c}^n + \frac{3}{2k}\beta_{m,2}^n\right) U_m^n + \left(\alpha_{m,+}^n + \frac{3}{2k}\beta_{m,3}^n\right) U_{m+1}^n = \\ &\quad \beta_{m,1}^n \left(f_{m-1}^n + \frac{2}{k}U_{m-1}^{n-1} - \frac{1}{2k}U_{m-1}^{n-2}\right) + \beta_{m,2}^n \left(f_m^n + \frac{2}{k}U_m^{n-1} - \frac{1}{2k}U_m^{n-2}\right) + \\ &\quad \beta_{m,3}^n \left(f_{m+1}^n + \frac{2}{k}U_{m+1}^{n-1} - \frac{1}{2k}U_{m+1}^{n-2}\right) = F_m^n \text{ (say)}, \quad m = 1, 2, \dots, M-1, \quad n = 2, 3, \dots, N, \quad (14) \end{aligned}$$

and

$$\begin{aligned} L_h^k U_m^n &\equiv \left( \alpha_{m,-}^n + \frac{1}{k} \beta_{m,1}^n \right) U_{m-1}^n + \left( \alpha_{m,c}^n + \frac{1}{k} \beta_{m,2}^n \right) U_m^n + \left( \alpha_{m,+}^n + \frac{1}{k} \beta_{m,3}^n \right) U_{m+1}^n = \\ &\quad \beta_{m,1}^n \left( f_{m-1}^n + \frac{1}{k} U_{m-1}^{n-1} \right) + \beta_{m,2}^n \left( f_m^n + \frac{1}{k} U_m^{n-1} \right) + \beta_{m,3}^n \left( f_{m+1}^n + \frac{1}{k} U_{m+1}^{n-1} \right) \\ &= F_m^n \text{ (say)}, \quad m = 1, 2, \dots, M-1, \quad n = 1. \end{aligned} \quad (15)$$

The scheme (14)-(15) can also be written as

$$A^n U^n = F^n, \quad n = 1, 2, \dots, N, \quad (16)$$

where

$$A^n = \begin{bmatrix} \alpha_{1,c}^n + \frac{3}{2k} \beta_{1,2}^n & \alpha_{1,+}^n + \frac{3}{2k} \beta_{1,3}^n & 0 & \dots & 0 \\ \alpha_{2,-}^n + \frac{3}{2k} \beta_{2,1}^n & \alpha_{2,c}^n + \frac{3}{2k} \beta_{2,2}^n & \alpha_{2,+}^n + \frac{3}{2k} \beta_{2,3}^n & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \alpha_{M-2,-}^n + \frac{3}{2k} \beta_{M-2,1}^n & \alpha_{M-2,c}^n + \frac{3}{2k} \beta_{M-2,2}^n & \alpha_{M-2,+}^n + \frac{3}{2k} \beta_{M-2,3}^n \\ 0 & \dots & 0 & \alpha_{M-1,-}^n + \frac{3}{2k} \beta_{M-1,1}^n & \alpha_{M-1,c}^n + \frac{3}{2k} \beta_{M-1,2}^n \end{bmatrix},$$

for  $n = 2, 3, \dots, N$  and

$$A^n = \begin{bmatrix} \alpha_{1,c}^n + \frac{1}{k} \beta_{1,2}^n & \alpha_{1,+}^n + \frac{1}{k} \beta_{1,3}^n & 0 & \dots & 0 \\ \alpha_{2,-}^n + \frac{1}{k} \beta_{2,1}^n & \alpha_{2,c}^n + \frac{1}{k} \beta_{2,2}^n & \alpha_{2,+}^n + \frac{1}{k} \beta_{2,3}^n & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \alpha_{M-2,-}^n + \frac{1}{k} \beta_{M-2,1}^n & \alpha_{M-2,c}^n + \frac{1}{k} \beta_{M-2,2}^n & \alpha_{M-2,+}^n + \frac{1}{k} \beta_{M-2,3}^n \\ 0 & \dots & 0 & \alpha_{M-1,-}^n + \frac{1}{k} \beta_{M-1,1}^n & \alpha_{M-1,c}^n + \frac{1}{k} \beta_{M-1,2}^n \end{bmatrix},$$

for  $n = 1$ .  $U^n = [U_1^n, U_2^n, \dots, U_{M-1}^n]^T$  and  $F^n = B^n g_1^n + g_2^n$ ,  $n = 1, 2, \dots, N$ , where

$$B^n = \begin{bmatrix} \beta_{1,2}^n & \beta_{1,3}^n & 0 & \dots & 0 \\ \beta_{2,1}^n & \beta_{2,2}^n & \beta_{2,3}^n & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \beta_{M-2,1}^n & \beta_{M-2,2}^n & \beta_{M-2,3}^n \\ 0 & \dots & 0 & \beta_{M-1,1}^n & \beta_{M-1,2}^n \end{bmatrix}, \quad n = 1, 2, \dots, N,$$

$$g_1^n = \begin{bmatrix} f_1^n + \frac{2}{k} U_1^{n-1} - \frac{1}{2k} U_1^{n-2} \\ f_2^n + \frac{2}{k} U_2^{n-1} - \frac{1}{2k} U_2^{n-2} \\ \vdots \\ f_{M-1}^n + \frac{2}{k} U_{M-1}^{n-1} - \frac{1}{2k} U_{M-1}^{n-2} \end{bmatrix}, \quad n = 2, 3, \dots, N, \quad g_1^n = \begin{bmatrix} f_1^n + \frac{1}{k} U_1^{n-1} \\ f_2^n + \frac{1}{k} U_2^{n-1} \\ \vdots \\ f_{M-1}^n + \frac{1}{k} U_{M-1}^{n-1} \end{bmatrix}, \quad n = 1,$$

$$g_2^n = \begin{bmatrix} \beta_{1,1}^n (f_0^n + \frac{2}{k}U_0^{n-1} - \frac{1}{2k}U_0^{n-2}) - (\alpha_{1,-}^n + \frac{3}{2k}\beta_{1,1}^n) U_0^n \\ 0 \\ \vdots \\ 0 \\ \beta_{M-1,3}^n (f_M^n + \frac{2}{k}U_M^{n-1} - \frac{1}{2k}U_M^{n-2}) - (\alpha_{M-1,+}^n + \frac{3}{2k}\beta_{M-1,3}^n) U_M^n \end{bmatrix}, \quad n = 2, 3, \dots, N,$$

$$g_2^n = \begin{bmatrix} \beta_{1,1}^n (f_0^n + \frac{1}{k}U_0^{n-1}) - (\alpha_{1,-}^n + \frac{1}{k}\beta_{1,1}^n) U_0^n \\ 0 \\ \vdots \\ 0 \\ \beta_{M-1,3}^n (f_M^n + \frac{1}{k}U_M^{n-1}) - (\alpha_{M-1,+}^n + \frac{1}{k}\beta_{M-1,3}^n) U_M^n \end{bmatrix}, \quad n = 1.$$

## 4 Error Analysis

**Lemma 4.1.** Assume that

$$\begin{aligned} & \frac{1}{2h}(3\beta_{m,1}^n a_{1,m-1}^n + \beta_{m,2}^n a_{1,m}^n - \beta_{m,3}^n a_{1,m+1}^n) + \beta_{m,1}^n \left(-a_{0,m-1}^n + \frac{3}{2k}\right) \\ & \leq \frac{1}{h^2}(\beta_{m,1}^n a_{2,m-1}^n + \beta_{m,2}^n a_{2,m}^n + \beta_{m,3}^n a_{2,m+1}^n), \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \frac{1}{2h}(\beta_{m,1}^n a_{1,m-1}^n - \beta_{m,2}^n a_{1,m}^n - 3\beta_{m,3}^n a_{1,m+1}^n) + \beta_{m,3}^n \left(-a_{0,m+1}^n + \frac{3}{2k}\right) \\ & \leq \frac{1}{h^2}(\beta_{m,1}^n a_{2,m-1}^n + \beta_{m,2}^n a_{2,m}^n + \beta_{m,3}^n a_{2,m+1}^n), \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N, \end{aligned} \quad (18)$$

then

$$\alpha_{m,-}^n + \frac{3}{2k}\beta_{m,1}^n \leq 0, \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N, \quad (19)$$

$$\alpha_{m,+}^n + \frac{3}{2k}\beta_{m,3}^n \leq 0, \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N. \quad (20)$$

*Proof.* The inequalities (19) and (20) can be easily proved using the values of the coefficients from (11) and (12) and the assumptions (17) and (18).  $\square$

**Lemma 4.2** (Discrete Maximum Principle). *Under the assumptions of the Lemma 4.1, the operator  $L_h^k$  defined by (14)-(15) satisfies discrete maximum principle. That is, if  $v_m^n$  and  $w_m^n$  are mesh functions that satisfy  $v_0^n \leq w_0^n$ ,  $v_M^n \leq w_M^n$  ( $n = 0, 1, \dots, N$ ),  $v_m^0 \leq w_m^0$ , ( $m = 0, 1, \dots, M$ ) and  $L_h^k v_m^n \leq L_h^k w_m^n$  ( $m = 1, 2, \dots, M-1$ ,  $n = 1, 2, \dots, N$ ), then  $v_m^n \leq w_m^n$  for all  $m, n$ .*

*Proof.* The coefficient matrix  $A^n$  corresponding to the linear system  $L_h^k U_m^n = F_m^n$  is a tridiagonal diagonally dominant matrix with non-positive off diagonal elements. Hence  $A^n$  is an irreducible M-matrix, which implies that it has positive inverse ([26, 16]). This proves the lemma.  $\square$

**Theorem 4.1.** *Let  $u$  be the solution of the continuous problem (6) and  $U_m^n$  be the solution of the discrete problem (9). Then under the assumptions (8),*

$$|u_m^n - U_m^n| \leq \mathcal{C}(k^2 + h^3), \quad m = 0, 1, \dots, M, \quad n = 0, 1, \dots, N, \quad (21)$$

where  $\mathcal{C}$  is a positive constant independent of  $k$  and  $h$ .

*Proof.* From (9a) we have

$$\begin{aligned} \mathcal{L}_h^k u_m^n - (Lu)_m^n &= \frac{\beta_{m,1}^n}{k} \left( \frac{3}{2} u_{m-1}^n - 2u_{m-1}^{n-1} + \frac{1}{2} u_{m-1}^{n-2} \right) + \frac{\beta_{m,2}^n}{k} \left( \frac{3}{2} u_m^n - 2u_m^{n-1} + \frac{1}{2} u_m^{n-2} \right) \\ &\quad + \frac{\beta_{m,3}^n}{k} \left( \frac{3}{2} u_{m+1}^n - 2u_{m+1}^{n-1} + \frac{1}{2} u_{m+1}^{n-2} \right) + [\alpha_{m,-}^n u_{m-1}^n + \alpha_{m,c}^n u_m^n + \alpha_{m,+}^n u_{m+1}^n] \\ &\quad - \beta_{m,1}^n f_{m-1}^n - \beta_{m,2}^n f_m^n - \beta_{m,3}^n f_{m+1}^n. \end{aligned} \quad (22)$$

Using (6a) and applying the Taylor's expansion in two variables, we have

$$\begin{aligned} |\mathcal{L}_h^k(u_m^n - U_m^n)| &= |\mathcal{L}_h^k u_m^n - (Lu)_m^n| \\ &\leq C_1 k^2 \int_0^1 \left[ \left| \frac{\partial^3 u}{\partial t^3}(x_m, t_n - ky) \right| + \left| \frac{\partial^3 u}{\partial t^3}(x_m, t_n - 2ky) \right| \right] dy \\ &\quad + C_2 h^3 \int_0^1 \left[ \left| \frac{\partial^4 u}{\partial x^3 \partial t}(x_m - hy, t_n) \right| + \left| \frac{\partial^4 u}{\partial x^3 \partial t}(x_m + hy, t_n) \right| \right. \\ &\quad \left. + \left| \frac{\partial^5 u}{\partial x^5}(x_m - hy, t_n) \right| + \left| \frac{\partial^5 u}{\partial x^5}(x_m + hy, t_n) \right| \right] dy, \\ &\leq C_3(k^2 + h^3), \quad m = 1, 2, \dots, M-1, \quad n = 2, 3, \dots, N. \end{aligned} \quad (23)$$

Here we have used the bounds given in (8). We have used backward Euler's method for the first time step and the one-step error of backward Euler's method is  $O(k^2)$  ([19]). Hence we have

$$|L_h^k(u_m^n - U_m^n)| \leq C_4(k^2 + h^3) \quad m = 1, 2, \dots, M-1, \quad n = 0, 1, \dots, N. \quad (24)$$

We construct the following barrier function

$$\varphi_m^n = \mathcal{C}(k^2 + h^3)(1 + t_n) \pm (u_m^n - U_m^n),$$

and apply the Lemma 4.2 on  $\varphi_m^n$ , to show

$$|u_m^n - U_m^n| \leq \mathcal{C}(k^2 + h^3) \text{ for every } m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N. \quad (25)$$

□

## 5 Numerical Experiments

Since the analytical solution of the generalized Black-Scholes equation is not known, we use the double mesh principle for computing the maximum error ( $E_{max}^{M,N}$ ), the root mean square error ( $E_{rms}^{M,N}$ ) and corresponding orders of convergence  $p_{max}^{M,N}$  and  $p_{rms}^{M,N}$  respectively through



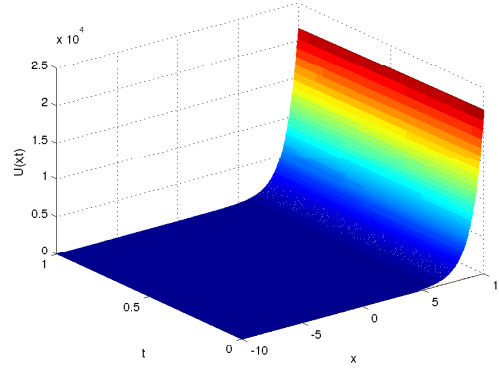


Figure 1: Computed solution for Example 5.1

the following formulas

$$E_{\max}^{M,N} = \max_{0 \leq m \leq M} |U^{M,N}(x_m, t_N) - U^{2M,4N}(x_{2m}, t_{4N})|,$$

$$E_{\text{rms}}^{M,N} = \sqrt{\frac{\sum_{m=0}^M [(U^{M,N}(x_m, t_N) - U^{2M,4N}(x_{2m}, t_{4N}))^2]}{M+1}},$$

$$p_{\max}^{M,N} = \log_2 \left( \frac{E_{\max}^{M,N}}{E_{\max}^{2M,4N}} \right) \text{ and } p_{\text{rms}}^{M,N} = \log_2 \left( \frac{E_{\text{rms}}^{M,N}}{E_{\text{rms}}^{2M,4N}} \right).$$

**Example 5.1.** Consider the initial boundary value problem (6) with  $\hat{\sigma}(x, t) = 0.3$ ,  $\hat{r}(x, t) = 0.04$ ,  $\hat{D}(x, t) = 0.02$ . Take  $\epsilon = 10^{-6}$ ,  $T = 1$ ,  $K = 0$ ,  $x_{\min} = -10$  and  $x_{\max} = 10$ .

M	10	$10 \times 2$	$10 \times 2^2$	$10 \times 2^3$	$10 \times 2^4$	$10 \times 2^5$
N	5	$5 \times 4$	$5 \times 4^2$	$5 \times 4^3$	$5 \times 4^4$	$5 \times 4^5$
$E_{\max}$		1.0409e01	1.1655e00	9.6519e-02	5.7905e-03	3.7861e-04
$p_{\max}$			3.1587	3.5940	4.0591	3.9349
$E_{\text{rms}}$		3.1585e00	2.6907e-01	1.9121e-02	1.2305e-03	7.7158e-05
$p_{\text{rms}}$			3.5532	3.8147	3.9579	3.9953

Table 1: Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence  $p_{\max}$  and  $p_{\text{rms}}$  for Example 5.1

**Example 5.2.** Consider the initial boundary value problem (6) with  $\hat{\sigma}(x, t) = 0.4(2 + (T - t) \sin(x))$ ,  $\hat{r}(x, t) = 0.06(1 + t \exp(-x))$ ,  $\hat{D}(x, t) = 0.02 \exp(-t - x)$ . Take  $\epsilon = 10^{-6}$ ,  $T = 1$ ,  $K = 0$ ,  $x_{\min} = -10$  and  $x_{\max} = 10$ .

The numerical solutions obtained by present scheme corresponding to Example 5.1 and Example 5.2 are shown in Fig. 1 and Fig. 2 respectively, the maximum errors and the root mean square errors and the corresponding orders of convergence are displayed in Table 1 and

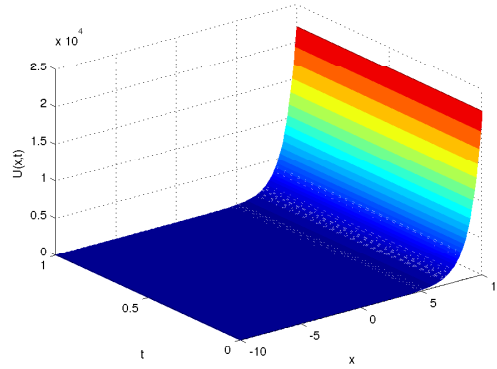


Figure 2: Computed solution for Example 5.2

M	10	$10 \times 2$	$10 \times 2^2$	$10 \times 2^3$	$10 \times 2^4$
N	5	$5 \times 4$	$5 \times 4^2$	$5 \times 4^3$	$5 \times 4^4$
$E_{\max}$		2.0738e02	1.4836e01	8.0790e-01	4.9512e-02
$p_{\max}$			3.8051	4.1989	4.0284
$E_{\text{rms}}$		6.2766e01	3.7972e00	2.1936e-01	1.3385e-02
$p_{\text{rms}}$			4.0469	4.1136	4.0346

Table 2: Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence  $p_{\max}$  and  $p_{\text{rms}}$  for Example 5.2

Table 2 respectively. Theorem 4.1 proves the third order convergence in space but the Tables 1 and 2 show fourth order convergence. These results give us a scope to improve the theoretical results.

## 6 Conclusion

In the present work we have developed a high order numerical scheme for pricing European call option governed by generalized Black-Scholes model. The scheme is constructed in the way that we can simultaneously discretize in space and time directions ([17]) which leads to simpler convergence analysis as also explained in [7]. Two step backward differentiation formula is used for temporal discretization and HODIE scheme with three nodal and symmetric auxiliary points is used for spacial discretization. Theoretically we have achieved  $O(k^2 + h^3)$  accuracy. But the numerical experiments show fourth order convergence in space which leads us to consider further improvement in the theoretical result.

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